

# Fair partitioning by straight lines

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## Abstract

A pizza is a pair of planar convex bodies  $A \subseteq B$ , where  $B$  represents the dough and  $A$  the topping of the pizza. A partition of a pizza by straight lines is a succession of double operations: a cut by a full straight line, followed by a Euclidean move of one of the resulting pieces; then the procedure is repeated. The final partition is said to be fair if each resulting slice has the same amount of  $A$  and the same amount of  $B$ . This note proves that, given an integer  $n \geq 2$ , there exists a fair partition by straight lines of any pizza  $(A, B)$  into  $n$  parts if and only if  $n$  is even. The proof uses the following result: For any planar convex bodies  $A, B$  with  $A \subseteq B$ , and any  $\alpha \in ]0, \frac{1}{2}[$ , there exists an  $\alpha$ -section of  $A$  which is a  $\beta$ -section of  $B$  for some  $\beta \geq \alpha$ . (An  $\alpha$ -section of  $A$  is a straight line cutting  $A$  into two parts, one of which has area  $\alpha|A|$ .) The question remains open if the word “planar” is dropped.

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Let  $\mathcal{K}$  denote the set of planar convex bodies, endowed with the usual Hausdorff-Pompeiu metric. The area of  $A \in \mathcal{K}$  is denoted by  $|A|$  and its boundary is denoted by  $\partial A$ . Following [6], what we call a *pizza* is a pair  $(A, B)$  of two nested planar convex bodies  $A \subseteq B \subset \mathbb{R}^2$ . We call  $A$  the *topping* and  $B$  the *dough*. Given a pizza  $(A, B)$  and an integer  $n \geq 2$ , a *fair partition of  $B$  in  $n$  slices* is a family of  $n$  internally disjoint convex subsets  $B_1, \dots, B_n$  such that

$$|B_1| = \dots = |B_n| \quad \text{and} \quad |A \cap B_1| = |A \cap B_2| = \dots = |A \cap B_n|.$$

For the sake of clarity, we call *pieces* the intermediate subsets and *slices* the final ones.

There is a wide literature upon the problem of fair partitioning a convex body, see e.g. [8]. The expressions “equipartition” and “balanced partition” are also used. If there is no other constraint than to obtain convex slices  $B_i$ , then it has been proven recently that the answer is positive for all  $n$ , see e.g. [7, 9, 10].

In [2] the authors use  $k$ -fans, which are half-lines starting from a common point. Since this process is very restrictive, the result is negative for  $k \geq 4$ .

Other rules have also been considered. One can ask to have same perimeter and same area for each slice, see e.g. [1, 5]. In [4], the author uses only cuts by horizontal and vertical segments.

In this note, we use a different cutting rule, which seems to be new: Divide  $B$  into two pieces with a straight cut. Each of the resulting pieces is a convex body, their interiors are disjoint, and their union is  $B$ . If  $B$  is divided into  $k$  pieces  $B_1, \dots, B_k$ , choose one of

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these pieces and divide it into two pieces with one straight cut. After  $n - 1$  cuts,  $B$  is divided into  $n$  convex slices. We will refer to this rule simply as to the CUTTING RULE, since no other rule is considered further in this note.

Our cutting rule is more restrictive than just partitioning  $B$  into  $n$  convex bodies. For example, a non-degenerate 3-fan partition cannot be obtained with our rule. However, for  $k \geq 4$ , our rule becomes somewhat less restrictive than a  $k$ -fan partition.

Before going further, we need to introduce some notation. The symbol  $\mathbb{S}^1$  stands for the standard unit circle,  $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ , endowed with its usual metric  $d(\theta, \theta') = \min\{|\tau - \tau'| ; \tau \in \theta, \tau' \in \theta'\}$ . Given  $\theta \in \mathbb{S}^1$ , let  $\vec{u}(\theta)$  denote the unit vector of direction  $\theta$ ,  $\vec{u}(\theta) = (\cos \theta, \sin \theta)$  and let  $\vec{u}'(\theta) = \frac{d\vec{u}}{d\theta}(\theta) = (-\sin \theta, \cos \theta)$ .

Given an oriented straight line  $\Delta$  in the plane,  $\Delta^+$  denotes the *closed* half-plane on the left bounded by  $\Delta$ , and  $\Delta^-$  is the closed half-plane on the right. We identify oriented straight lines with points of the cylinder  $\mathbf{C} = \mathbb{S}^1 \times \mathbb{R}$ , associating each pair  $(\theta, t) \in \mathbf{C}$  to the line oriented by  $\vec{u}(\theta)$  and passing at the signed distance  $t$  from the origin. In other words, the half-plane  $\Delta^+$  is given by  $\Delta^+ = \{x \in \mathbb{R}^2 ; \langle x, \vec{u}'(\theta) \rangle \geq t\}$ . We endow  $\mathbf{C}$  with the natural distance  $d((\theta, t), (\theta', t')) = (d(\theta, \theta')^2 + |t - t'|^2)^{1/2}$ . The reason to introduce the space  $\mathbf{C}$  is the following: several times throughout the paper we will say that some oriented line moves continuously. The continuity will always refer to the topology of  $\mathbf{C}$ .

Given  $\alpha \in ]0, 1[$  and  $A \in \mathcal{K}$ , an  $\alpha$ -section of  $A$  is an oriented line  $\Delta$  such that  $|\Delta^- \cap A| = \alpha|A|$ . For all  $\alpha \in ]0, 1[$  and all  $\theta \in [0, 2\pi[$ , there exists a unique  $\alpha$ -section of  $A$  of direction  $\theta$ , denoted by  $\Delta(\alpha, \theta, A)$ . The line  $\Delta(\alpha, \theta, A)$ , treated as a function, depends continuously on its three arguments.

Our first result has been conjectured in [6].

**Theorem 1.** *For any planar convex bodies  $A, B$  with  $A \subset B$ , and any  $\alpha \in ]0, \frac{1}{2}[$ , there exists an  $\alpha$ -section of  $A$  which is a  $\beta$ -section of  $B$  for some  $\beta \geq \alpha$ .*

*Proof.* By contradiction, if every  $\alpha$ -section  $\Delta(\alpha, \theta, A)$  of  $A$  is a  $\beta(\theta)$ -section of  $B$  with  $\beta(\theta) < \alpha$  then, by continuity of  $\theta \mapsto \beta(\theta)$  and by compactness of  $\mathbb{S}^1$ , there exists  $\varepsilon > 0$  such that, for all  $\theta \in \mathbb{S}^1$ ,  $\beta(\theta) \leq \alpha - \varepsilon$ .

Choose an integer  $n > \frac{1}{\varepsilon}$ . Choose  $x_0 \in \partial A$  arbitrarily and, for each positive integer  $i \leq n$ , define  $x_i$  recursively by  $x_i \in \partial A$  and the oriented line  $D_i = (x_{i-1}x_i)$  is an  $\alpha$ -section of  $A$ .

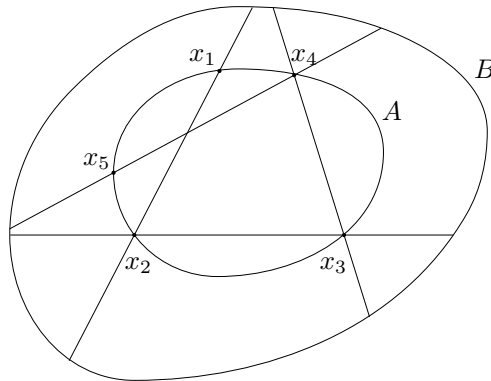


Figure 1: Construction of consecutive  $\alpha$ -sections

We call a cap of  $B$  the intersection  $D_i^- \cap B$  and a cap of  $A$  the intersection  $D_i^- \cap A$ . We thus have  $n + 1$  points on  $\partial A$  and  $n$  caps of  $A$ , resp.  $B$ . For each  $x \in B$ , let  $K(x)$  be the number of caps of  $B$  which cover  $x$ , i.e.  $K(x) = \text{card}\{i \in \{1, \dots, n\} ; x \in D_i^-\}$ .

Let  $k = \min\{K(x) ; x \in \partial A\}$ ; it is the number of complete tours made by  $x_0, \dots, x_n$ . Observe that, for all  $x \in \partial A$  we have  $k \leq K(x) \leq k + 1$  and that, for each  $0 < m \leq n$ , the arc  $\widehat{x_{m-1}x_m} = \partial A \cap D_m^-$  contains  $K(x_m) + 1$  or  $K(x_m) + 2$  points among  $x_0, \dots, x_n$  (including  $x_{m-1}$  and  $x_m$ ). This comes from the fact that, if  $x_i$  is on the arc  $\widehat{x_{m-1}x_m}$ , then  $x_m$  is on the arc  $\widehat{x_i x_{i+1}}$ .

We claim that  $K(x) \leq k + 1$  for all  $x \in A$ . Indeed, let  $x \in A$  and consider a cap  $A_m$  containing  $x$  (if  $x$  belongs to no cap, there is nothing to prove). If another cap  $A_i$  contains  $x$ , then  $x_{i-1}$  or  $x_i$  must belong to the open arc  $\widehat{x_{m-1}x_m} \setminus \{x_{m-1}, x_m\}$ . Now for each of the points  $x_j$  on this open arc, at most one of the caps  $A_j$  or  $A_{j+1}$  can contain  $x$ , hence  $x$  is in at most  $k + 1$  caps. It follows that the sum of areas of all caps  $D_i^- \cap A$  is at most  $(k + 1)|A|$ , hence  $n\alpha \leq k + 1$ .

Also, for all  $x \in B \setminus A$ , we have  $K(x) \geq k$ . Hence  $\sum_{i=1}^n |D_i^- \cap (B \setminus A)| \geq k|B \setminus A|$ . Thus there exists  $i_0$  such that  $|D_{i_0}^- \cap (B \setminus A)| \geq \frac{k}{n}|B \setminus A| \geq (\alpha - \frac{1}{n})|B \setminus A|$ . It follows that  $|D_{i_0}^- \cap B| = |D_{i_0}^- \cap A| + |D_{i_0}^- \cap (B \setminus A)| \geq \alpha|A| + (\alpha - \frac{1}{n})|B \setminus A| \geq (\alpha - \frac{1}{n})|B|$ , i.e.  $D_{i_0}$  is a  $\beta$ -section of  $B$ , with  $\beta \geq (\alpha - \frac{1}{n}) > \alpha - \varepsilon$ , a contradiction.  $\square$

**Remark 2.** A question whether Theorem 1 extends to an arbitrary dimension remains open. However, one can show that for every  $d > 2$  there exists a constant  $\alpha_0(d) > 0$  such that the  $d$ -dimensional analogue of Theorem 1 holds for all  $\alpha \in ]0, \alpha_0[$ . The idea is similar to the 2-dimensional proof, but instead of an  $n$ -fold covering of  $\partial A$  by caps we use a 1-fold covering, namely, the *economic cap covering*, defined, for example, in [3]. However, this method is not very efficient, giving only a very small value of  $\alpha_0$ . Hence we leave the details of the proof to the reader.

Another equivalent formulation of Theorem 1, which will be more convenient, is as follows. The proof of the equivalence is easy and left to the reader.

**Corollary 3.** *For any planar convex bodies  $A, B$  with  $A \subset B$ , and any  $\alpha \in ]0, \frac{1}{2}[$ , there exists an  $\alpha$ -section of  $B$  which is a  $\beta$ -section of  $A$  for some  $\beta \leq \alpha$ .*

Our next result, Theorem 4, concerns a fair pizza partition problem using the CUTTING RULE. It has been already mentioned in [6] as a consequence of Theorem 1, but without a proof of implication. Here we give a proof, and thus confirm the result.

**Theorem 4.** *Let  $n$  be a positive integer. Then*

1. *If  $n$  is even, then for every pair  $A \subseteq B$  of nested planar convex bodies there exists a fair partition obeying the CUTTING RULE.*
2. *If  $n$  is odd, then for some pairs  $A \subseteq B$  such a partition may not exist.*

*Proof.* It is easy to check that two concentric disks  $A$  and  $B$  cannot be divided in a fair way into an odd number of slices: The first cut divides the pizza in two pieces, containing  $k$ , resp.  $l$  final slices, with  $k + l = n$  odd, hence  $k \neq l$ , and the smaller piece will not have enough topping.

To construct a fair partition for all even  $n$ , we proceed by induction.

For  $n = 2$ , this follows from the intermediate value theorem. Given  $n \in \mathbb{N}$ ,  $n$  even  $\geq 4$ , and a pair of nested convex bodies  $A \subseteq B$ , assume that a fair partitioning exists for any pair of nested convex bodies and any even integer  $i < n$ .

If  $n = 4k$ , then the intermediate value theorem yields a fair partitioning of two equal halves, and, by induction hypothesis, each of these halves admits a fair partitioning in  $2k$  slices.

Let  $n = 4k + 2$ . Then we set  $\alpha = \frac{2k}{4k+2}$  and consider two subcases.

1. Suppose that we can cut  $B$  into two convex pieces  $B_1$  and  $B_2$  of areas  $|B_1| = \alpha|B|$ ,  $|B_2| = (1 - \alpha)|B|$  so that  $|A \cap B_1| = \alpha|A|$ ,  $|A \cap B_2| = (1 - \alpha)|A|$ .

Then, by induction,  $B_1$  and  $B_2$  have both a fair partitioning in  $2k$ , resp.  $2k+2$ , slices, and this gives a fair partitioning of  $B$  in  $n$  slices.

**2.** If we are not in subcase **1** then no  $\alpha$ -section of  $B$  contains an  $\alpha$ -portion of  $A$ . Then from Corollary 3 it follows that each  $\alpha$ -section of  $B$  (with this  $\alpha$ ) is a  $\beta$ -section of  $A$  for some  $\beta < \alpha$ .

Cut  $B$  into two fair halves  $B'$  and  $B''$ . We claim that there is a cut of  $B'$  (and, similarly, of  $B''$ ) such that it produces a slice of area  $\frac{1}{n}|B|$  with the topping part of area  $\frac{1}{n}|A|$  (i.e., a fair slice).

Consider a piece  $C_1 \subset B'$  between two parallel lines, one of which is the initial cut, and the other one is chosen so that  $|C_1| = \frac{1}{n}|B|$ . By the construction,  $B' \setminus C_1$  is an  $\alpha$ -section of  $B$ , so  $|A \cap (B' \setminus C_1)| < \alpha|A|$  and hence  $|A \cap C_1| > (\frac{1}{2} - \alpha)|A| = \frac{1}{n}|A|$ .

On the other hand, by Corollary 3, there exists a  $\frac{2}{n}$ -section of  $B'$ , which is at most  $\frac{2}{n}$ -section of  $A \cap B'$ . If  $C_2$  is the piece of  $B'$  obtained by that section, then  $|C_2| = \frac{1}{n}|B|$ , and  $|A \cap C_2| \leq \frac{1}{n}|A|$ .

Using the intermediate value theorem for  $\frac{2}{n}$ -sections of  $B'$ , we obtain that there is a slice  $C$ , which is cut from  $B'$  by a single line, such that  $|C| = \frac{1}{n}|B|$ , and  $|A \cap C| = \frac{1}{n}|A|$ .

By induction hypothesis, the piece  $B' \setminus C$  admits a fair partition into  $2k$  slices. As a result, there is a fair partition of  $B'$  into  $2k+1$  slices. The same can be done for  $B''$ , yielding a fair partition of the whole pizza.

□

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